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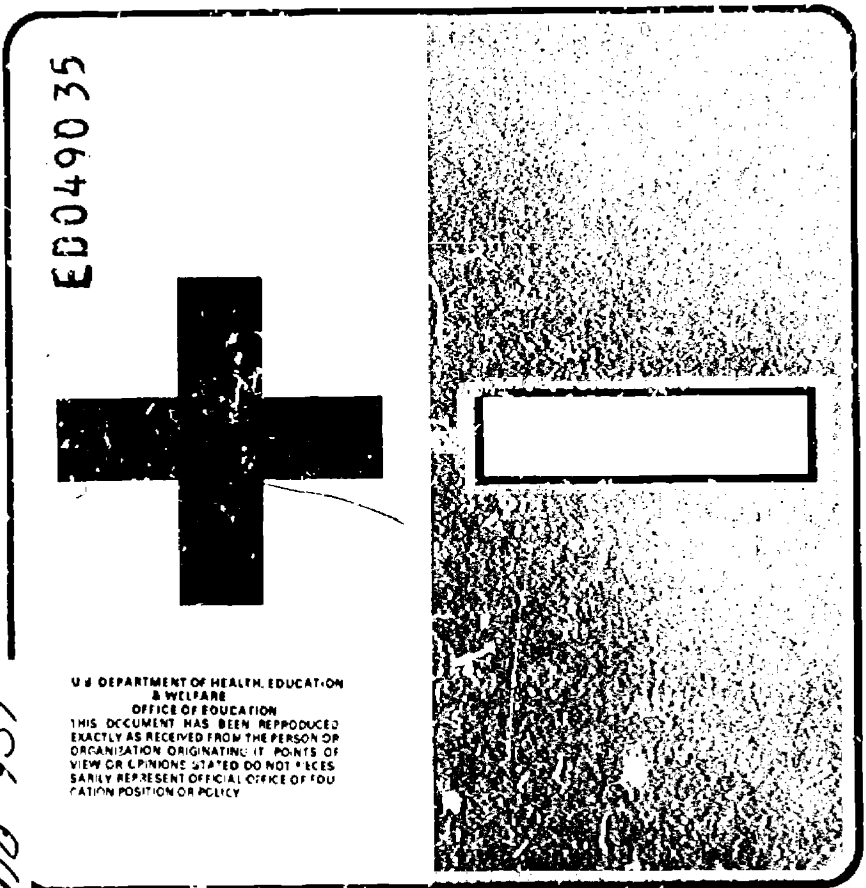
ABSTRACT

This is the ninth in a series of ten self-contained units designed for use by students in ninth grade general mathematics classes. This unit is divided into six sections dealing with different concepts involving positive and negative numbers. Some concepts presented include positive and negative integers, addition and subtraction, multiplication, absolute value, and rational numbers. Though the topics are standard they are dealt with in non-traditional methods emphasizing discovery learning. Many exercises, diagrams, and topics for discussion are included. (CT)

9 Positive and Negative Numbers

EXPERIENCES IN MATHEMATICAL DISCOVERY



National Council of Teachers of Mathematics

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UNIT NINE OF

Experiences in Mathematical Discovery

Positive and Negative Numbers

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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Preface

"Experiences in Mathematical Discovery" is a series of ten self-contained units, each of which is designed for use by students of ninth-grade general mathematics. These booklets are the culmination of work undertaken as part of the General Mathematics Writing Project of the National Council of Teachers of Mathematics (NCTM).

The titles in the series are as follows:

Unit 1: *Formulas, Graphs, and Patterns*

Unit 2: *Properties of Operations with Numbers*

Unit 3: *Mathematical Sentences*

Unit 4: *Geometry*

Unit 5: *Arrangements and Selections*

Unit 6: *Mathematical Thinking*

Unit 7: *Rational Numbers*

Unit 8: *Decimals, Ratios, Percents*

Unit 9: *Positive and Negative Numbers*

Unit 10: *Measurement*

This project is experimental Teachers may use as many units as suit their purposes. Authors are encouraged to develop similar approaches to the topics treated here, and to other topics, since the aim of the NCTM in making these units available is to stimulate the development of special materials that can be effectively used with students of general mathematics.

Preliminary versions of the units were produced by a writing team that met at the University of Oregon during the summer of 1963. The units were subsequently tried out in ninth-grade general mathematics classes throughout the United States.

Oscar F. Schaaf, of the University of Oregon, was director of the 1963 summer writing team that produced the preliminary materials. The work of planning the content of the various units was undertaken by Thomas J. Hill, Oklahoma City Public Schools, Oklahoma City, Oklahoma; Paul S. Jorgensen, Carleton College, Northfield, Minnesota; Kenneth P. Kidd, University of Florida, Gainesville, Florida; and Max Peters, George W. Wingate High School, Brooklyn, New York.

The Advisory Committee for the General Mathematics Writing Project was composed of Emil J. Berger (*chairman*), St. Paul Public Schools, St. Paul, Minnesota; Irving Adler, North Bennington, Vermont; Stanley J. Bezuszka, S. J., Boston College, Chestnut Hill, Massachusetts; Eugene P. Smith, Wayne State University, Detroit, Michigan; and Max A. Sobel, Montclair State College, Upper Montclair, New Jersey.

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Finally, a word of grateful thanks is extended to the NCTM headquarters staff for their assistance, and in particular to Clare Stifft and Gladys Majette for their careful attention to production editing details.

EMIL J. BERGER
Chairman, Advisory Committee
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Contents

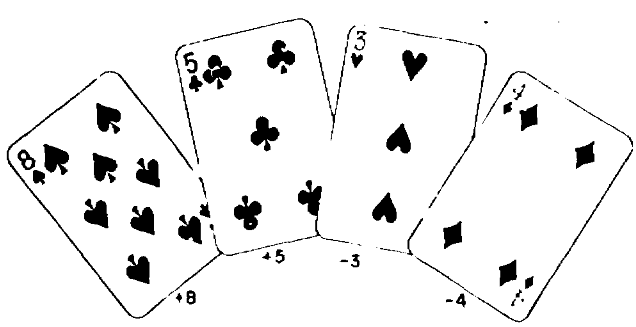
Positive and Negative Numbers

1 Touchdown

John and David made up a new card game called TOUCHDOWN. To play the game they used an ordinary deck of cards with the picture cards removed. They let each red card stand for a *loss* of as many points as the number on the card, and each black card for a *gain* of as many points as the number on the card. Each ace, regardless of color, was considered a gain of 15 points.

John shuffled the deck and dealt David four cards. David's cards were the 8 of spades, the 5 of clubs, the 3 of hearts, and the 4 of diamonds. Notice that two cards were from black suits and two were from red suits.

John said, "You have gains of 8 and 5, and losses of 3 and 4."



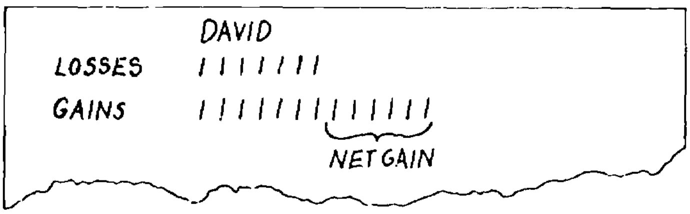
The boys decided to represent the net result in the following way:

$$(^{+}8) + (^{+}5) + (^{-}3) + (^{-}4)$$

John did some paper work and said, "That gives you a net gain of 6."

"How did you do that so fast?" asked David.

John showed David the work that he did on paper.



"I tried to match gains with losses, and right away I saw that I had six marks for gains left over," replied John.

"You could also get the answer by subtracting 7 from 13," said David. "Since the difference is 6 and you had more gains than losses, you would know that you had a net gain of 6."

David wrote $^{+}6$ on his scorecard, shuffled the deck, and dealt John four cards. John's cards were the 9 of hearts, the 5 of hearts, the 6 of spades, and the 7 of spades. John wrote $^{-}1$ on his scorecard. This meant that he had a net loss of 1 point. How do you think John determined this loss?

Class Discussion 1

1. What is the greatest net gain a player can have on any one hand?
2. What is the greatest net loss a player can have on any one hand?
3. Using S, C, H, and D to stand for spades, clubs, hearts, and diamonds respectively, what would be the net gain or loss for a hand consisting of 10S, 7C, 8H, 9D?
4. In playing the game of TOUCHDOWN, each player totals

his score after each hand. After several hands David's and John's scorecards looked like those shown below.

David	
Hand Score	Total Score
$^{+}6$	$^{+}6$
$^{+}12$	$^{+}18$
$^{-}5$	$^{+}13$
$^{-}15$	$^{-}2$

John	
Hand Score	Total Score
$^{-}1$	$^{-}1$
$^{+}8$	$^{+}7$
$^{-}3$	$^{+}4$
$^{-}2$	$^{+}2$

a. At the point in the game when David's total score was $^{-}2$ and John's total score was $^{+}2$, which boy was ahead?

b. Later in the game John had a total score of $^{+}15$ and David had a total score of $^{-}19$. Who was ahead?

5. There are two ways to win the game of TOUCHDOWN. A player wins if he gets a total score of $^{+}30$ or more. He also wins if his opponent gets a score of $^{-}15$ or less. Is a score of $^{-}19$ less than a score of $^{-}15$?

6. a. If you combine two losses, is the result a loss?

b. If you combine two gains, is the result a gain?

c. When would a gain and a loss cancel each other, so that the player would break even? How would you represent a break-even score?

d. What results are possible when you combine a gain and a loss?

7. The numbers you have been using to describe gains and losses (and breaking even) are called *integers*. The set of integers is listed below.

$$\{\cdots, ^{-}3, ^{-}2, ^{-}1, 0, ^{+}1, ^{+}2, ^{+}3, \cdots\}$$

The symbol "$^{-}3$" is read "negative three." The symbol "$^{+}3$" is read "positive three." And the symbol "0" is read "zero."

a. Which integer would you use to indicate that you broke even?

b. The set of integers is listed in a special way. What tells you that $^{-}4$, $^{-}5$, and $^{-}6$ are members of the set of integers?

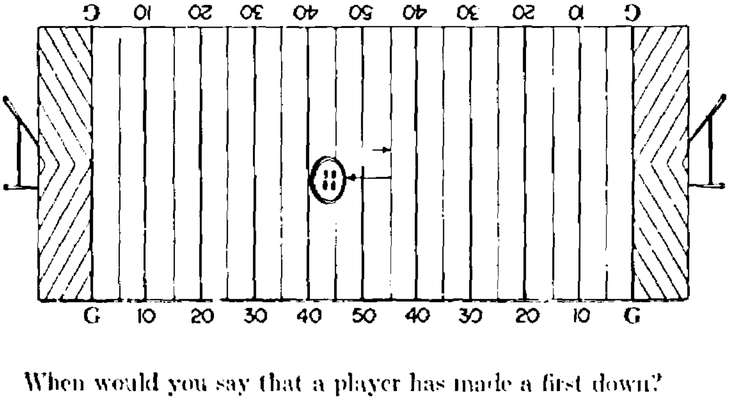
c. What tells you that $^{+}4$, $^{+}5$, and $^{+}6$ are members of the set of integers?

A Football Game

You can change the game of TOUCHDOWN to make it a FOOTBALL game. Draw a diagram of a football field on a large piece of cardboard. Draw lines to represent the yard markers and boundaries. Place a button on the field to represent the ball. Make one move for each hand of cards.

Use the same deck of cards that you used for TOUCHDOWN. Let each hand of four cards represent four downs. Let each red card stand for a loss of as many yards as the number on the card, and each black card for a gain of as many yards as the number on the card. Let each ace stand for a completed forward pass that is good for a gain of 15 yards. If you like, put the Jacks in the deck of cards, and let each Jack stand for a 5-yard penalty.

To start the game, flip a coin to see who kicks off (deals the cards). Assume that the kick-off is always received on the receiver's 40-yard line. Play a game of FOOTBALL. Play four 10-minute quarters.



When would you say that a player has made a first down?
When would a player lose the ball on downs?
When would you say that a player has made a touchdown?
If your opponent has the "ball," when would you score a safety?

8. The set of integers can be separated into the following subsets:

Negative Integers	$\{^-1, ^-2, \cdots\}$
Positive Integers	$\{^+1, ^+2, \cdots\}$
Zero	$\{0\}$

a. Which subset is used to represent losses?
b. Which subset is used to represent breaking even?
c. Which subset is used to represent gains?

9. a. In the game of TOUCHDOWN, does a loss result in a lower or a higher score than breaking even?
b. Does a gain result in a lower or a higher score than breaking even?
c. Do you think each negative integer is less than zero?
d. Do you think each positive integer is greater than zero?

10. The symbol "$<$" means "is less than," and the symbol "$>$" means "is greater than." Translate each English sentence into a mathematical sentence. Hint: Rewrite each sentence using only mathematical symbols.
a. Negative four is less than zero.
b. Positive one is greater than zero.
c. Zero is greater than negative ten.
d. Negative ten is less than negative three.
e. Positive fifteen is greater than positive ten.
f. Positive three is greater than negative one.

11. Which sentences in exercise 10 are true?

12. Using gains and losses as scores in the game of TOUCHDOWN, can you think of a way to decide which of two integers is greater and which is less than the other?
a. Does a loss of 10 points result in a lower score than a loss of 5 points? Is it true that $^-10 < ^-5$?
b. Does a loss of 3 points result in a lower score than no loss? Is it true that $^-3 < 0$?
c. Does every loss result in a lower score than a gain? Is it true that $^-1 < ^+2$?
d. Is zero greater than every negative integer? Is it true that $0 > ^-2$?

e. Is zero less than every positive integer? Is it true that $0 < {}^{+}3$?

f. Read the sentence below. Is the sentence true?

$$^{-}1 < 0 < {}^{+}1.$$

g. Is the following sentence true?

$$\cdots {}^{-}5 < {}^{-}2 < {}^{-}1 < 0 < {}^{+}1 < {}^{+}3 \cdots .$$

13. Combining gains and losses suggests the idea of addition. Suppose that a football player gains 7 yards on one play and loses 10 yards on the next play. The result is a net loss of 3 yards for the two plays. This idea can be expressed by using the mathematical sentence below.

$$^{+}7 + {}^{-}10 = {}^{-}3.$$

This sentence is read "Positive seven plus negative ten equals negative three." Write a mathematical sentence for each of the following:

a. A loss of 5 yards followed by a gain of 9 yards results in a net gain of 4 yards.

b. A gain of 2 yards followed by a loss of 2 yards results in a net yardage of 0.

c. A gain of 5 yards followed by a gain of 3 yards results in a net gain of 8 yards.

d. A loss of 4 yards followed by a loss of 2 yards results in a net loss of 6 yards.

Gains and losses suggest the idea of positive and negative numbers. Gains suggest positive numbers. Losses suggest negative numbers. Breaking even suggests a number that is neither positive nor negative. This number is zero.

The set of all positive numbers, negative numbers, and zero includes a special set called the set of integers.

Using gains and losses as scores in a game suggests a way of telling which of two integers is greater and which is less than the other. Since a gain results in a higher score than a loss, any positive integer is greater than any negative integer. Furthermore, zero is greater than any negative integer and less than any positive integer.

The sentence below tells you the order of the integers in relation to each other.

$$\cdots {}^{-}4 < {}^{-}3 < {}^{-}2 < {}^{-}1 < 0 < {}^{+}1 < {}^{+}2 < {}^{+}3 < {}^{+}4 \cdots .$$

Combining gains and losses suggests a way of determining what the sum of two integers should be.

1. Integers can be used to express ideas other than gains and losses. What integer is suggested by each idea expressed below?
   a. The temperature outdoors is 32 degrees below zero.
   b. You make a withdrawal of $20 from the bank.
   c. Jim's score on a mathematics test is 10 points above the class average.
   d. The city of Denver is 5,280 feet above sea level.
   e. The countdown was stopped 10 seconds before blast-off.
   f. You deposit $17 in a savings account and withdraw $15.
   g. Death Valley is 282 feet below sea level.
   h. The elevation of New York City is said to be "sea level."
2. Use an integer to describe each of the following situations.
   a. The temperature is 25 degrees below zero.
   b. Peter spent $8.
   c. Carol earned $10.
   d. Marge paid a dental bill of $45.
   e. Ray received a royalty check for $721.
   f. The airplane cruised at an altitude of 35 000 feet.
   g. The submarine dived to a depth of 3,000 feet.
   h. The stock market average was down 3 points.
3. a. Following a loss of 9 points, what would you need to break even?
   b. Following a gain of 4 points, what would you need to break even?

c. For any loss is there exactly one gain that will cancel the loss so that you will break even?
d. For any gain is there exactly one loss that will cancel the gain so that you will break even?

4. Bookkeepers sometimes record the amount of money paid for expenses in red ink and the amount of money received in black ink.
 a. If this scheme is used, how can you tell if there are more expenses than receipts?
 b. How can you tell if there are more receipts than expenses?
 c. What do you think the expression "in the red" means? What does "in the black" mean?
 d. What does it mean to say "the books balance"?

5. Translate each of the following into a mathematical sentence.
 a. A gain of 8 followed by a gain of 5 results in a net gain of 13.
 b. A loss of 4 followed by a loss of 7 results in a net loss of 11.
 c. A loss of 5 followed by a gain of 5 results in breaking even.
 d. A loss of 10 followed by a gain of 3 results in a net loss of 7.
 e. A loss of 5 followed by a gain of 11 results in a net gain of 6.

6. Complete each sentence so that the resulting sentence is true.
 a. $^{+}3 + ^{-}7 =$ ____. c. $^{+}13 + ^{-}14 =$ ____.
 b. $^{-}10 + ^{+}11 =$ ____. d. $^{-}7 + ^{+}7 =$ ____.

7. Complete each sentence using whichever symbol, $=$, $<$, or $>$, makes the sentence true.
 a. $^{-}25 + ^{+}15$ ____ $^{-}10$. c. $^{-}4$ ____ $^{+}2 + ^{-}5$.
 b. $^{+}3 + ^{-}2$ ____ 0.

8. For each integer, what integer should you add so that the sum will be zero?
 a. $^{+}17$ b. $^{-}12$ c. $^{+}99$ d. 0

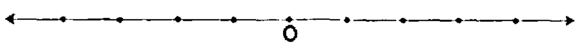
9. If the sum of two integers is zero, then each integer is the *opposite* of the other.
 a. If an integer is positive, what kind of integer is its opposite?
 b. If an integer is negative, what kind of integer is its opposite?
 c. What is the opposite of zero?

2 Trips along a Line

Integer is my name.
A line is my nation.
A point is my dwelling place.
I am a destination.

This poem may seem like a pu- ' . It suggests that integers can be assigned to points in a line. The poem also suggests that if you take a trip along a line, your destination will be a point to which an integer has been assigned.

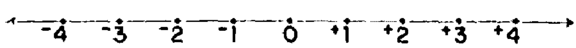
Shown below is a line that has been divided into segments of the same length. Each segment is *one unit* long. The integer 0 has been assigned to one of the division points. This is the usual way to start assigning integers to points in a line.



In the next diagram, positive and negative integers have been assigned to points in a line according to the following order.

$$\cdots {}^-4 < {}^-3 < {}^-2 < {}^+1 < 0 < {}^+1 < {}^+2 < {}^+3 < {}^+4 \cdots$$

Such a *matching* of points and numbers is called a *number line.*

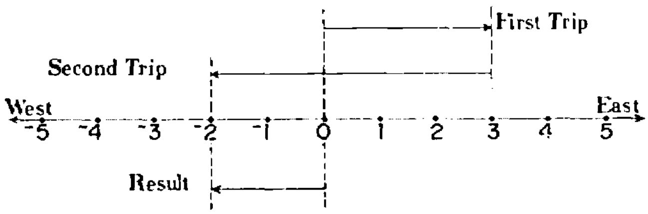


Since the positive integers and zero can be matched with the same points as the familiar set of whole numbers {0, 1, 2, $\cdots$ }, we can use the positive integers and zero in the same way that we use whole numbers. Therefore, from now on we will write 3 in place of $^+3$, 7 in place of $^+7$, and so on.

Class Discussion 2

1. As you go to the right along the number line, do numbers become larger or smaller? What happens as you go to the left?
2. Look at the diagram below. The two segments above the line show a pair of trips. The arrow tips indicate direction. The

first trip starts at 0, and the second trip starts where the first trip ends.



a. Is the direction of the first trip east or west?
b. How many units long is the first trip?
c. What is the direction of the second trip?
d. How many units long is the second trip?
e. Now think of a single trip that starts where the first trip starts and ends where the second trip ends. What is the direction and length of this trip? We will call this new trip the *result* of the first and second trips combined.

3. You can find the result of two consecutive trips along the number line by counting unit segments in the direction of the first trip and then counting unit segments in the direction of the second trip. Below is a shorthand way of describing two consecutive trips and the result.

$$\overrightarrow{3} + \overleftarrow{5} = \overleftarrow{2}.$$

The sentence is read: "A trip of three units east followed by a trip of five units west has the result of a trip of two units west."
Translate each sentence into English.

a. $\overleftarrow{8} + \overrightarrow{10} = \overrightarrow{2}$. **c.** $\overrightarrow{9} + \overrightarrow{2} = \overrightarrow{11}$.
b. $\overrightarrow{5} + \overleftarrow{6} = \overleftarrow{1}$. **d.** $\overleftarrow{15} + \overleftarrow{17} = \overleftarrow{32}$.

4. In each exercise, find the missing trip.

a. $\overrightarrow{13}$ + ____ = $\overrightarrow{5}$. **c.** $\overrightarrow{100} + \overleftarrow{100}$ = ____.
b. ____ + $\overleftarrow{8}$ = $\overleftarrow{12}$. **d.** $\overrightarrow{16}$ + ____ = $\overrightarrow{19}$.

5. a. What will be the final destination point if you take a trip of 8 units east and follow it by a trip of 5 units west?
   b. Suppose you first took a trip of 5 units west and then followed it with a trip of 8 units east. Would you arrive at the same destination?
   c. Do you think reversing the order of two trips will change the result?

6. Decide whether or not each sentence is true.

   a. $\overrightarrow{2} + \overleftarrow{8} = \overleftarrow{8} + \overrightarrow{2}$.

   b. $\overleftarrow{3} + \overleftarrow{11} = \overleftarrow{11} + \overleftarrow{3}$.

   c. $\overrightarrow{15} + \overrightarrow{22} = \overrightarrow{22} + \overrightarrow{15}$.

7. You can find the result of three trips by combining the result of two trips with the third trip. Complete each sentence below. Compare the final results.

   a. $(\overrightarrow{5} + \overrightarrow{3}) + \overleftarrow{12} = \overrightarrow{8} + \overleftarrow{12} =$ ____.

   b. $\overrightarrow{5} + (\overrightarrow{3} + \overleftarrow{12}) = \overrightarrow{5} + \overleftarrow{9} =$ ____.

8. Complete each sentence below. Compare the final results.

   a. $(\overleftarrow{2} + \overrightarrow{10}) + \overleftarrow{7} = \overrightarrow{8} + \overleftarrow{7} =$ ____.

   b. $\overleftarrow{2} + (\overrightarrow{10} + \overleftarrow{7}) = \overleftarrow{2} + \overrightarrow{3} =$ ____.

9. Does changing the grouping of three trips change the final result?

10. The trip symbolized by $\dot{0}$ is called the zero trip.
    a. What is the starting point of the zero trip?
    b. What is the endpoint of the zero trip?

11. In each exercise, find the missing trip.

    a. $\dot{0} + \overrightarrow{4} =$ ____.

    b. $\overrightarrow{6} + \dot{0} =$ ____.

    c. $\overrightarrow{10} + \dot{0} =$ ____.

12. If the zero trip is combined with any other trip, what is the result?

13. In each exercise, find the missing trip.

   a. $\overrightarrow{4} + \overleftarrow{4} =$ ____.

   b. $\overleftarrow{7} + \overrightarrow{7} =$ ____.

   c. $\dot{0} + \dot{0} =$ ____.

   d. $\overrightarrow{2} +$ ____ $= \dot{0}$.

   e. ____ $+ \overleftarrow{6} = \dot{0}$.

14. Suppose $\dot{0}$ is the result of two trips.
   a. Do the two trips have the same length?
   b. Do they have opposite directions?

15. Two trips that have the same length and opposite directions are referred to as "opposite trips." Give the opposite trip for each of the following.
   a. $\overrightarrow{6}$ **b.** $\overleftarrow{9}$ **c.** $\overrightarrow{10}$

[illegible]

In the Class Discussion exercises above you saw how to assign integers to points in a line. You also saw how to take trips along a line.

Each trip that starts at zero corresponds to exactly one integer, and each integer corresponds to exactly one trip that starts at zero. The table below illustrates the idea.

You can find the sum of any two integers by using the number line and the idea of combining trips. The idea of combining trips is related to adding integers. The properties of addition of integers are summarized below.

1. The sum of any two integers is an integer.
   If x and y represent integers, then $x + y$ represents an integer.

2. The order in which two integers are added does not affect the sum.

   If x and y represent integers, then $x + y = y + x$.

3. The way in which three integers are grouped when adding does not affect the sum.

   If x, y, and z represent integers, then

   $$x + (y + z) = (x + y) + z.$$

4. Adding zero to a given integer results in the given integer.

   If x represents an integer, then $x + 0 = 0 + x = x$.

5. Each integer has exactly one opposite.

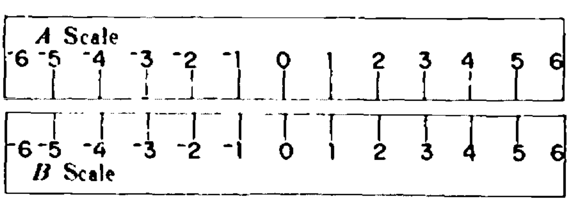
   If x represents an integer, then there is exactly one integer y which when added to x gives zero.

   Also, if $x + y = 0$, then x is the opposite of y, and y is the opposite of x.

   Zero is its own opposite; that is, $0 + 0 = 0$.

Exercises—2

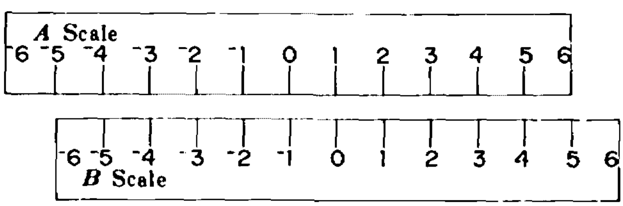
1. In this exercise you are asked to make a slide rule for adding integers.
   a. Cut out two rectangular strips of cardboard as shown in the diagram below.
   b. Mark off one edge of each strip into segments one-half inch long.
   c. Label the marks on the edges of the rectangular strips with integers as shown below.



2. Now let's use the slide rule to find the sum $1 + {}^-4$.

a. Since 1 is the first number of the pair to be added, move the *B* scale to the right until the 0 on the *B* scale is below 1 on the *A* scale.

b. Since $^-4$ is the second number of the pair to be added, locate $^-4$ on the *B* scale. Directly above $^-4$ on the *B* scale is $^-3$ on the *A* scale. Thus, $^-3$ is the sum of 1 and $^-4$.



3. a. Would you have to change the slide rule setting shown in exercise 2 to find any of the following sums?

$1 + {}^-1 \qquad 1 + 1$

$1 + {}^-2 \qquad 1 + 2$

$1 + {}^-3 \qquad 1 + 3$

$1 + {}^-4 \qquad 1 + 4$

$1 + {}^-5 \qquad 1 + 5$

$1 + {}^-6 \qquad 1 + 0$

b. Find each sum in exercise 3a.

c. Can you find the following sums with the slide rule shown in exercise 2 if you move the *B* scale to a different setting? Explain your answer.

$1 + 6 \qquad 1 + {}^-7$

$1 + 7 \qquad 1 + {}^-8$

d. Use your slide rule to find each sum. This may take a little practice, but don't give up.

$5 + {}^-5 \qquad 3 + 2 \qquad {}^-3 + 2$

${}^-2 + {}^-2 \qquad {}^-4 + 5 \qquad 5 + {}^-4$

4. a. Use your slide rule to find each sum.

$^{-}1 + {}^{-}1$ $\quad$ $^{-}1 + {}^{-}4$ $\quad$ $^{-}3 + {}^{-}2$

$^{-}1 + {}^{-}2$ $\quad$ $^{-}5 + {}^{-}1$ $\quad$ $^{-}4 + {}^{-}2$

$^{-}3 + {}^{-}1$ $\quad$ $^{-}2 + {}^{-}2$ $\quad$ $^{-}3 + {}^{-}3$

b. Is each sum a negative integer?

c. Do you think the sum of two negative integers is always a negative integer?

5. a. Use your slide rule to find each sum.

$1 + 1$ $\quad$ $4 + 1$ $\quad$ $3 + 2$

$2 + 1$ $\quad$ $1 + 5$ $\quad$ $4 + 2$

$1 + 3$ $\quad$ $2 + 2$ $\quad$ $3 + 3$

b. Is each sum a positive integer?

c. Do you think that the sum of two positive integers is always a positive integer?

6. a. Use your slide rule to find each sum.

$^{-}2 + 5$ $\quad$ $3 + {}^{-}1$ $\quad$ $6 + {}^{-}4$

$^{-}5 + 1$ $\quad$ $^{-}3 + 3$ $\quad$ $^{-}6 + 6$

$^{-}4 + 3$

b. If you add a positive integer and a negative integer, the sum can be ________, ________, or ________.

7. a. Use your slide rule to find each sum.

$0 + 4$ $\quad$ $^{-}3 + 0$ $\quad$ $6 + 0$

$3 + 0$ $\quad$ $0 + 2$ $\quad$ $^{-}4 + 0$

b. Explain how your slide rule can be used to show that adding zero to a given integer results in the given integer.

8. The slide rule you used for adding integers is really a simple computer. The diagram in figure 1 represents a computer for adding integers that is even simpler. This computer is called a *nomograph*. It has three parallel scales, *A*, *B*, and *C*.

Each scale is perpendicular to a zero line. The *A* and *B* scales are the same distance from the *C* scale. The length of the unit segment on the *A* scale is the same as on the *B* scale. The unit segment on the *C* scale is half as long as on the *A* and *B* scales. Make a tracing of the nomograph or make an enlarged copy.

Nomograph for Adding Integers

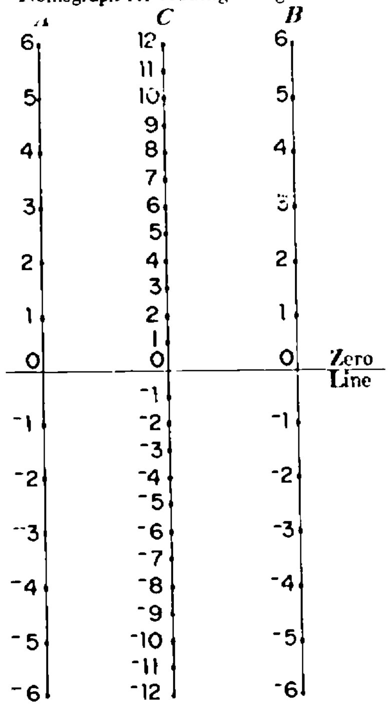


Fig. 1

9. Printed at the right of the nomograph shown in figure 2 is the mathematical sentence $4 + {}^{-}6 = {}^{-}2$.

Nomograph for Adding Integers

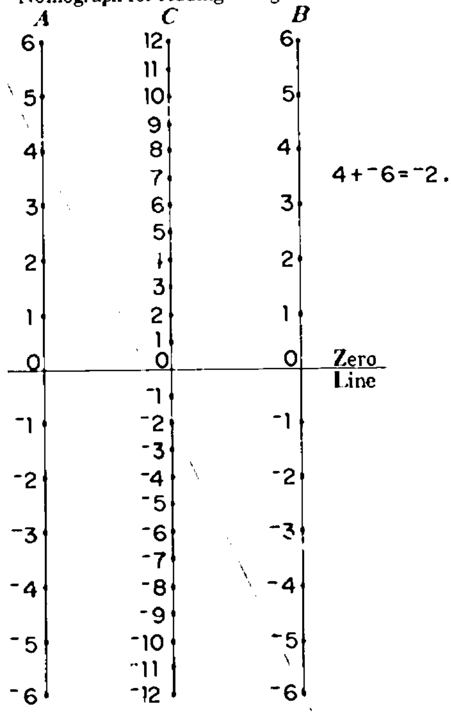


Fig. 2

The red line crosses the *A* scale at 4, the *B* scale at ⁻6, and the *C* scale at ⁻2. Is ⁻2 the sum of 4 and ⁻6?

a. Find each sum below, using a nomograph. Use the edge of a ruler to line up points on the *A* and *B* scales. The sum will be on the *C* scale.

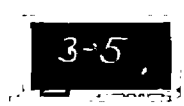
⁻6 + ⁻6	5 + ⁻5	6 + 4
⁻6 + ⁻1	⁻4 + ⁻4	5 + ⁻3
⁻5 + ⁻4	0 + 0	4 + ⁻6
2 + 4	2 + 2	3 + 0

b. Complete each sentence.
  (1) The sum of two negative integers is a ______ integer.
  (2) The sum of two positive integers is a ______ integer.
  (3) The sum of a positive integer and a negative integer is ______, ______, or ______.
  (4) The sum of an integer and its opposite is ______.
  (5) The sum of a given integer and ______ is the given integer.

10. Do the positive integers and zero behave like the whole numbers when adding?

11. By experimenting with the nomograph, try to discover how to subtract integers.

3 Subtracting by Adding



Miss Thomas wrote "3 − 5" on the chalkboard. The students laughed because they thought Miss Thomas had made a mistake. Alice, one of the students who sat near the chalkboard, exclaimed quickly, "You can't subtract 5 from 3!"

"Why not?" asked Miss Thomas.

"Because 5 is greater than 3," replied Alice.

"Well, can you subtract 3 from 5?" asked Miss Thomas.

"Yes," said Alice, "The answer is 2."

"How do you know that 5 minus 3 equals 2?" asked Miss Thomas.

"Because 2 plus 3 equals 5," said Alice.

"Then," said Miss Thomas, "all you have to do to find the difference 3 − 5 is to find a number which when added to 5 gives 2.

"But there is no such number," pleaded Alice.

"Let's look at the nomograph," said Miss Thomas.

The nomograph in figure 3 shows that $^-2 + 5 = 3$ is a true sentence. The red line crosses the A scale at $^-2$, the B scale at 5, and the C scale at 3.

Nomograph for Adding Integers

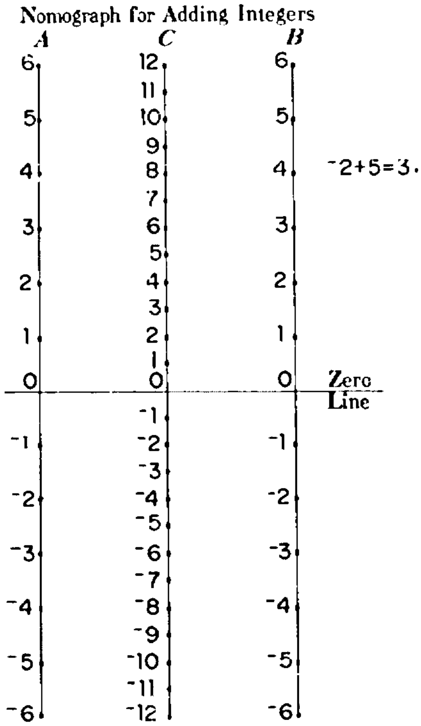


Fig. 3

Now cover up the A scale with your hand. This is like asking "What number plus 5 equals 3?"

$$\_\_\_ + 5 = 3.$$

Or, what is the same thing, "3 minus 5 equals what number?"

$$3 - 5 = \_\_\_.$$

The answer to each question is $^-2$. Using Alice's argument,

$$3 - 5 = {}^-2 \quad \text{because} \quad {}^-2 + 5 = 3.$$

Class Discussion 3

1. Complete each sentence.
   a. $^-2 - 6 =$ ___ because ___ $+ 6 = {}^-2$.
   b. $3 - 4 =$ ___ because ___ $+ 4 = 3$.
   c. $^-1 - {}^-3 =$ ___ because ___ $+ {}^-3 = {}^-1$.
   d. ___ $- {}^-4 = {}^-3$ because $^-3 + {}^-4 =$ ___.
   e. $5 -$ ___ $= {}^-6$ because $^-6 +$ ___ $= 5$.
2. The two mathematical sentences below are said to be *related.* The first is a subtraction sentence. The second is an addition sentence.

$$^-2 - 6 = 4.$$
$$4 + {}^-6 = {}^-2.$$

   Write the related addition sentence for each subtraction sentence.
   a. $2 - 3 = {}^-1$
   b. $^-3 - 2 = x$.
   c. $4 - {}^-5 = x$.
   d. $^-5 - {}^-6 = x$.
3. Suppose you want to find the difference of $^-3$ and 2.
   a. Begin by writing a subtraction sentence that has $^-3 - 2$ on one side.
   b. Write the related addition sentence.
   c. Solve the related addition sentence.
   d. Check your solution using a nomograph. If your solution is correct, it is the difference of $^-3$ and 2.
4. Suppose you want to find the difference of 5 and $^-2$. Follow the steps below.
   a. Write a subtraction sentence.
   b. Write the related addition sentence.
   c. Solve the addition sentence.
   d. Check the solution of the addition sentence using a nomograph. If your solution is correct, it is the difference of 5 and $^-2$.
5. a. What is the sum of 4 and $^-4$?
   b. Is the sum of an integer and its opposite always zero?
   c. What is the sum of any given integer and zero?
   d. Is the sum of a given integer and zero ever zero?

6. Use the ideas in exercise 5 to find the difference of 5 and ${}^{-}4$.

a. Follow steps (1) through (7).

(1) Let x represent the difference.	x
(2) Write a subtraction sentence.	$5 - {}^{-}4 = x.$
(3) Write the related addition sentence.	$x + {}^{-}4 = 5.$
(4) Add the opposite of ${}^{-}4$ to each side.	$(x + {}^{-}4) + 4 = 5 + 4.$
(5) Group ${}^{-}4$ with 4.	$x + ({}^{-}4 + 4) = 5 + 4.$
(6) Find the sum ${}^{-}4 + 4$.	$x + 0 = 5 + 4.$
(7) Find the sum $x + 0$.	$x = 5 + 4.$

b. Compare the open sentences in steps (2) and (7):

$$5 - {}^{-}4 = x \qquad \text{and} \qquad x = 5 + 4.$$

c. Do you agree that $5 - {}^{-}4 = 5 + 4$?

d. Do you agree that $5 - {}^{-}4 = 9$?

7. Does *subtracting an integer have the same effect as adding its opposite?*

8. Explain what the title of Section 3 means.

9. Which of the following sentences are true?

a. $9 - 5 = 9 + {}^{-}5.$

b. $4 - 1 = 4 + {}^{-}1.$

c. $7 - 6 = 7 + {}^{-}6.$

10. Suppose you have ten marbles and lose five.

a. Express this idea as the difference of two integers.

b. Express the same idea as the sum of two integers.

c. Does subtracting a positive integer have the same effect as adding its opposite?

11. Suppose you are penalized five yards in a football game. You can use the integer ${}^{-}5$ to describe this situation. Suppose next that your opponent declines the penalty. This is like taking away a five-yard penalty.

a. Express the above idea as the difference of two integers.

b. Express the same idea as the sum of two integers.

c. Does subtracting a negative integer have the same effect as adding its opposite?

Summary—3

1. When working with whole numbers you *cannot* subtract a larger number from a smaller number.
2. With integers, however, you can subtract a larger number from a smaller number. For example, the difference $2 - 7$ equals $^-5$ because $^-5 + 7 = 2$.
3. One way to find the difference $2 - 7$ is to change $2 - 7$ to $2 + {}^-7$ and then add. It is because we can do this that the present section has the title "Subtracting by Adding."
4. Subtraction of one integer from another is always possible. If x and y represent integers, then $x - y$ represents an integer.
5. Recall that if the order in which two integers are added is changed, the sum is still the same. For example, $2 + {}^-3 = {}^-1$, and $^-3 + 2 = {}^-1$. Subtraction of integers does *not* have this property. Only one example is needed to show this: $5 - {}^-1 = 6$, but $^-1 - 5 = {}^-6$. When the order of two integers in a subtraction is changed, the differences are *opposites* of each other.
6. Recall that when adding, the way in which three integers are grouped does not affect the sum. Subtraction of integers does not have this property. The two mathematical sentences below are sufficient to show this.

$$2 - (5 - 8) = 2 - (^-3) = 5.$$
$$(2 - 5) - 8 = (^-3) - 8 = {}^-11.$$

7. Slide rules and nomographs can be used to add integers and to subtract integers. Of course, if you want to subtract 579 from 2,471 you will need an awfully big slide rule or nomograph.

Exercises—3

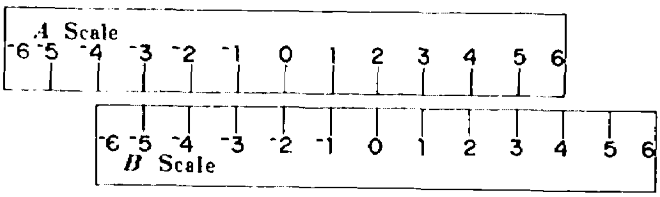
1. Use any method you know to find the following differences.

a. $5 - 8$	**d.** $11 - 8$	**g.** $18 - {}^-2$
b. $^-6 - 2$	**e.** $14 - 20$	**h.** $^-12 - {}^-12$
c. $4 - 10$	**f.** $6 - {}^-6$	**i.** $^-21 - 21$

j. $17 - 18$
k. $43 - {}^-43$
l. ${}^-43 - 43$
m. ${}^-43 - {}^-43$
n. $27 - 56$
o. $27 - {}^-56$
p. ${}^-27 - {}^-56$
q. ${}^-14 - 0$
r. $35 - 0$
s. $13 - {}^-20$
t. ${}^-20 - 13$

2. Find the difference ${}^-4 - 2$ using the slide rule that you made for adding integers.

   a. Move the *B* scale so that 0 is below 2 on the *A* scale.
   b. Look for ${}^-4$ on the *A* scale.
   c. Read the difference (answer) from the *B* scale directly below ${}^-4$ on the *A* scale.



   d. Is your answer ${}^-6$?
   e. Is it true that ${}^-4 - 2 = {}^-6$? How would you check?
   f. Is it true that ${}^-6 + 2 = {}^-4$? Check your answer by using the slide rule.

3. Find each difference using the slide rule. Check each answer by adding.

a. ${}^-5 - {}^-6$
b. ${}^-4 - {}^-5$
c. ${}^-2 - {}^-4$
d. $2 - {}^-3$
e. ${}^-3 - 2$
f. $4 - {}^-2$
g. ${}^-5 - {}^-3$
h. ${}^-5 - {}^-5$
i. $0 - {}^-3$
j. $5 - 6$
k. $2 - 3$
l. $3 - 5$
m. $6 - 5$
n. ${}^-5 - 1$
o. ${}^-4 - 2$
p. ${}^-2 - 2$
q. ${}^-2 - 1$
r. ${}^-2 - {}^-3$
s. ${}^-5 - {}^-2$
t. ${}^-6 - {}^-5$
u. $1 - {}^-5$
v. ${}^-1 - {}^-4$
w. ${}^-4 - {}^-4$
x. ${}^-6 - {}^-4$
y. ${}^-2 - {}^-5$
z. ${}^-3 - {}^-6$

4 Looking for Patterns, Stretching, and Reversing Trips

You may not recognize it from the title, but this section is concerned with multiplication of integers.

Recall that the positive integers and zero behave like whole numbers when adding. The same is true when multiplying. Look at the "2" table at the right. A dot (·) has been used to show multiplication. The part printed in black is a multiplication table of 2's for whole numbers. This part is also a multiplication table of 2's for positive integers and zero. The red part involves multiplication of negative integers.

We will call the number before the dot the *first factor* and the number after the dot the *second factor*. The number following the equals symbol "=" is the *product*.

Look for a pattern in the table. Notice that the first factor is the same in each line. The second factor decreases by 1 with each line. What happens to the product?

"2" Table
$2 \cdot 9 = 18$
$2 \cdot 8 = 16$
$2 \cdot 7 = 14$
$2 \cdot 6 = 12$
$2 \cdot 5 = 10$
$2 \cdot 4 = 8$
$2 \cdot 3 = 6$
$2 \cdot 2 = 4$
$2 \cdot 1 = 2$
$2 \cdot 0 = 0$
$2 \cdot {}^{-}1 = {}^{-}2$
$2 \cdot {}^{-}2 = {}^{-}4$
$2 \cdot {}^{-}3 = {}^{-}6$
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---- ----
---- ----
---- ----
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Class Discussion 4a

1. Suppose you continue the pattern started in the "2" table. The next line will be

$$2 \cdot {}^{-}4 = {}^{-}8.$$

   a. What will the next line be? And the line after that?
   b. Is 2 the first factor in each line?
   c. Does the product decrease by 2 with each line?
   d. Do you think the product of 2 and a negative integer is always a negative integer?

2. Make a "3" table. Use 3 as the first factor. The first two lines of your table should be

$$3 \cdot 9 = 27,$$
$$3 \cdot 8 = 24.$$

   a. Does the product decrease by 3 with each line?
   b. Do you think the product of 3 and a negative integer is always a negative integer?

3. Make a "4" table.

4. Make a "5" table.

5. In the tables you made, is the product of a positive integer and a negative integer always a negative integer?

6. Another "2" table is started at the right. In this table, 2 is the second factor in each line.
   a. Describe the pattern of products.
   b. Do products in this table decrease by 2 in the same way as in the first "2" table?
   c. Do you think a negative integer times a positive integer always equals a negative integer?

7. Make a "3" table having 3 as the second factor.

8. Make a "4" table having 4 as the second factor.

9. Make a "5" table having 5 as the second factor.

Another "2" Table
$9 \cdot 2 = 18$
$8 \cdot 2 = 16$
$7 \cdot 2 = 14$
$6 \cdot 2 = 12$
$5 \cdot 2 = 10$
$4 \cdot 2 = 8$
$3 \cdot 2 = 6$
$2 \cdot 2 = 4$
$1 \cdot 2 = 2$
$0 \cdot 2 = 0$
$^-1 \cdot 2 = {}^-2$
$^-2 \cdot 2 = {}^-4$
$^-3 \cdot 2 = {}^-6$
$^-4 \cdot 2 = {}^-8$
$^-5 \cdot 2 = {}^-10$

Because positive integers behave like whole numbers when multiplying, you know that the product of two positive integers is always a positive integer.

On the other hand, the pattern of products in each table shows that a positive integer times a negative integer is a negative integer. The order of the factors does not matter.

Class Discussion 4b

1. Look for a pattern in the "$^{-}2$" table at the right.
   a. Do the products increase by 2 with each line?
   b. Continue the pattern of products on the blank lines.
   c. Does the pattern suggest that the product of two negative integers is positive?
2. Make a "$^{-}3$" table.
3. Make a "$^{-}4$" table.
4. Make a "$^{-}5$" table.

"$^{-}2$" Table

$^{-}2 \cdot 9 = {}^{-}18$
$^{-}2 \cdot 8 = {}^{-}16$
$^{-}2 \cdot 7 = {}^{-}14$
$^{-}2 \cdot 6 = {}^{-}12$
$^{-}2 \cdot 5 = {}^{-}10$
$^{-}2 \cdot 4 = {}^{-}8$
$^{-}2 \cdot 3 = {}^{-}6$
$^{-}2 \cdot 2 = {}^{-}4$
$^{-}2 \cdot 1 = {}^{-}2$
$^{-}2 \cdot 0 = 0$
$^{-}2 \cdot {}^{-}1 = 2$

Summary—4b

From Summary 4a you know that –

1. The product of a positive integer times a positive integer is a positive integer.
2. The product of a positive integer times a negative integer is a negative integer. The order of the factors does not matter.

The Class Discussion exercises above should have convinced you that –

3. The product of a negative integer times a negative integer is a positive integer.

4. The product of any given integer and zero is zero.
5. The product of any given integer and one is the given integer.

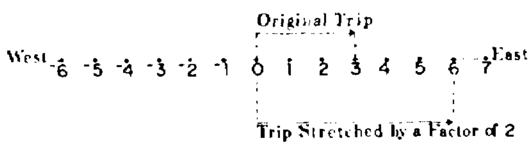
Class Discussion **4c**

Let us turn again to the idea of taking trips along a number line. However, this time we shall consider only trips that start at zero. With this agreement, let $\overrightarrow{3}$ symbolize a trip of 3 units east that starts at zero.

1. Multiplying by a positive integer greater than 1 can be thought of as stretching a trip. For example, $2 \cdot 3$ can be thought of as

$$2 \cdot \overrightarrow{3}.$$

The first factor 2 tells you to stretch a trip of 3 units east to twice its length. The diagram illustrates the idea.



a. Is it true that $2 \cdot \overrightarrow{3} = \overrightarrow{6}$?
b. Does $\overrightarrow{6}$ represent a trip of 6 units east?

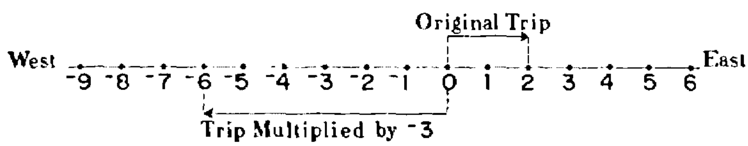
2. What would multiplying by a factor of 2 do to a trip of 3 units west?
a. Make a diagram to show your answer.
b. What is the direction of the resulting trip?
c. What is the length of the resulting trip?
d. $2 \cdot \overleftarrow{3} =$ ____.
e. $2 \cdot {}^{-}3 =$ ____.

3. Multiplying by $^{-}1$ can be thought of as reversing the direction of a trip.
a. Make a diagram that shows the trip $\overrightarrow{2}$ and also the trip indicated by $^{-}1 \cdot \overrightarrow{2}$.
b. Make a diagram that shows the trip $\overleftarrow{3}$ and also the trip indicated by $^{-}1 \cdot \overleftarrow{3}$.

4. Multiplying by $^{-}3$ can be thought of as reversing the direction of a trip and stretching it to three times its length. For example, $^{-}3 \cdot 2$ can be interpreted as

$$^{-}3 \cdot \overrightarrow{2}.$$

The diagram illustrates the idea.



a. What is the direction of the resulting trip? Is this opposite the direction of $\overrightarrow{2}$?
b. What is the length of the resulting trip?
c. Write a symbol for the resulting trip.
d. $^{-}3 \cdot 2 =$ ____.

5. Make a diagram to illustrate that

$$^{-}4 \cdot \overleftarrow{2} = \overrightarrow{8}.$$

6. Decide which sentences are true and which are false.

a. $^{-}4 \cdot \overrightarrow{5} = \overrightarrow{20}$.
b. $^{-}4 \cdot \overleftarrow{5} = \overrightarrow{20}$.
c. $6 \cdot \overleftarrow{3} = \overrightarrow{18}$.
d. $^{-}6 \cdot \overrightarrow{3} = \overleftarrow{18}$.
e. $0 \cdot \overrightarrow{10} = \dot{0}$.
f. $10 \cdot \overrightarrow{0} = \overrightarrow{10}$.
g. $4 \cdot \dot{0} = \overleftarrow{24}$.

7. Make each sentence true.

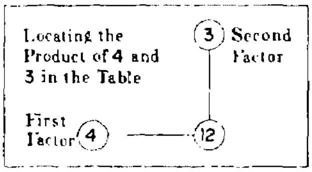
a. ____ $\cdot \overrightarrow{7} = \overleftarrow{21}$.
b. $^{-}5 \cdot$ ____ $= \overleftarrow{25}$.
c. $4 \cdot \overleftarrow{8} =$ ____.
d. $^{-}1 \cdot \overrightarrow{7} =$ ____.
e. $^{-}1 \cdot \overleftarrow{9} =$ ____.

8. Make each sentence true.

a. $4 \cdot (2 \cdot {}^{-}\dot{3}) =$ ____.
b. $(^{-}2 \cdot 3) \cdot {}^{-}\overrightarrow{4} =$ ____.
c. $(7 \cdot {}^{-}2) \cdot$ ____ $= \dot{14}$
d. $(^{-}2 \cdot {}^{-}5) \cdot$ ____ $= \dot{0}$.

Exercises—4a

1. A multiplication table for integers is started in figure 4. Copy and complete the unfinished table. Follow steps 1a and 1b.
   - **a.** Find the pattern in each row that is started. Complete the rows that are started.
   - **b.** Find the pattern in each column. Complete the columns.



Multiplication Table for Integers

×	9	8	7	6	5	4	3	2	1	0	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$^{-}5$	$^{-}6$	$^{-}7$	$^{-}8$	$^{-}9$
9																			
8																			
7																			
6				36	30	24	18	12	6	0	$^{-}6$	12							
5				30	25	20	15	10	5	0	$^{-}5$	10							
4				24	20	16	12	8	4	0	$^{-}4$	$^{-}8$							
3				18	15	12	9	6	3	0	$^{-}3$	$^{-}6$							
2				12	10	8	6	4	2	0	$^{-}2$	$^{-}4$							
1				6	5	4	3	2	1	0	$^{-}1$	$^{-}2$							
0				0	0	0	0	0	0	0	0	0							
$^{-}1$				$^{-}6$	$^{-}5$	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2							
$^{-}2$																			
$^{-}3$																			
$^{-}4$																			
$^{-}5$																			
$^{-}6$																			
$^{-}7$																			
$^{-}8$																			
$^{-}9$																			

Fig. 4

2. Use the table you completed to answer the following:
   a. What result do you get when you multiply any integer by 0?
   b. What result do you get when you multiply a given integer by 1?
   c. ${}^{-}5 \cdot {}^{-}6 =$ ____.
   d. ${}^{-}6 \cdot {}^{-}5 =$ ____.
   e. Does changing the order of the factors when multiplying change the result?

3. Multiply inside the parentheses first. Then multiply by the number outside.
   a. $({}^{-}2 \cdot 3) \cdot {}^{-}5$
   b. ${}^{-}2 \cdot (3 \cdot {}^{-}5)$

4. Multiply inside the parentheses first. Then multiply by the number outside.
   a. $6 \cdot ({}^{-}4 \cdot {}^{-}3)$
   b. $(6 \cdot {}^{-}4) \cdot {}^{-}3$

5. a. Did you get the same result in exercises 3a and 3b? In exercises 4a and 4b?
   b. Do you think changing the grouping of three integers when multiplying changes the result? Or does it leave the result unchanged?

6. Find each product.

a. $7 \cdot {}^{-}2$	k. $25 \cdot {}^{-}4$
b. ${}^{-}3 \cdot 8$	l. ${}^{-}25 \cdot {}^{-}4$
c. ${}^{-}5 \cdot {}^{-}6$	m. $16 \cdot {}^{-}15$
d. $14 \cdot {}^{-}5$	n. ${}^{-}18 \cdot {}^{-}18$
e. ${}^{-}12 \cdot {}^{-}12$	o. ${}^{-}18 \cdot 18$
f. $17 \cdot {}^{-}2$	p. $21 \cdot {}^{-}4$
g. ${}^{-}13 \cdot 10$	q. ${}^{-}15 \cdot {}^{-}11$
h. ${}^{-}15 \cdot 0$	r. ${}^{-}11 \cdot {}^{-}15$
i. $0 \cdot {}^{-}156$	s. ${}^{-}24 \cdot {}^{-}10$
j. ${}^{-}25 \cdot 4$	t. ${}^{-}10 \cdot {}^{-}24$

7. Make each sentence true.

a. $5 \cdot$ ___ $= {}^{-}20$.	c. ___ $\cdot {}^{-}3 = 15$.	e. ${}^{-}4 \cdot$ ___ $= 32$.
b. ${}^{-}8 \cdot 2 =$ ___.	d. $6 \cdot {}^{-}12 =$ ___.	f. ___ $\cdot 11 = 0$.

Exercises—4b

1. Division and multiplication of integers are related in the same way as division and multiplication of whole numbers. For example,

$$12 \div 3 = 4 \qquad \text{because} \qquad 4 \cdot 3 = 12.$$

Here is a scheme for using the relation between division and multiplication of integers to find quotients. Find the diagram below in the multiplication table for integers.

This part of the table tells you that

$$5 \cdot {}^{-}2 = {}^{-}10.$$

It also tells you that

$${}^{-}10 \div {}^{-}2 = 5,$$

and that

$${}^{-}10 \div 5 = {}^{-}2.$$

2. Look at the multiplication table for integers which you completed. Find $^{-}18$ in the body of the table. How many times does $^{-}18$ appear in the body of the table? Check to see if each sentence is true.

a. ${}^{-}18 \div 6 = {}^{-}3.$
b. ${}^{-}18 \div {}^{-}6 = 3.$
c. ${}^{-}18 \div 3 = {}^{-}6.$
d. ${}^{-}18 \div {}^{-}3 = 6$
e. ${}^{-}18 \div 9 = {}^{-}2.$
f. ${}^{-}18 \div {}^{-}9 = 2.$
g. ${}^{-}18 \div 2 = {}^{-}9.$
h. ${}^{-}18 \div {}^{-}2 = 9.$

3. Make each sentence true.

a. ${}^{-}18 \div {}^{-}2 = 9$ because $9 \cdot {}^{-}2 =$ ____.
b. $27 \div {}^{-}3 = {}^{-}9$ because ____ $\cdot\, {}^{-}3 = 27.$
c. ${}^{-}25 \div {}^{-}5 = 5$ because ____ $\cdot\, {}^{-}5 = {}^{-}25.$
d. $15 \div {}^{-}5 = {}^{-}3$ because ${}^{-}3 \cdot$ ____ $= 15.$

e. ____ $\div {}^-2 = {}^-5$ because ${}^-5 \cdot {}^-2 =$ ____.
f. ${}^-14 \div {}^-7 =$ ____ because ____ $\cdot {}^-7 = {}^-14$.
g. ${}^-16 \div {}^-8 =$ ____ because ____ $\cdot {}^-8 = {}^-16$.
h. ${}^-30 - {}^-6 =$ ____ because ____ ${}^-6 = {}^-30$.

4. a. Does $7 \div 3$ equal an integer?
b. By what integer can you multiply 3 to get 7?
c. Does ${}^-9 \div 2$ equal an integer?
d. By what integer can you multiply 2 to get ${}^-9$?
e. If you divide a given integer by another integer do you always get an integer for the quotient?

5. Trying to find an integer x which makes it true that

$$5 \div 0 = x$$

is like trying to solve the multiplication sentence

$$x \cdot 0 = 5.$$

a. Is there a replacement for x which makes the multiplication sentence true?

b. Is it possible to divide 5 by 0?

c. If an integer is different from zero, is it possible to divide this integer by 0?

6. a. The multiplication sentences on the left below can be obtained from the multiplication table for integers. Try to complete the related division sentences at the right.

$1 \cdot 0 = 0.$ $\qquad 0 \div 0 = 1.$

${}^-7 \cdot 0 = 0.$ $\qquad 0 \div 0 = {}^-7.$

$3 \cdot 0 = 0.$ $\qquad 0 \div 0 =$ ____.

${}^-8 \cdot 0 = 0.$ $\qquad 0 \div 0 =$ ____.

b. Do you see that $0 \div 0$ does not represent a *definite* number?

Division of any number different from zero by 0 is impossible, and $0 \div 0$ does not represent a definite number. For these reasons division by 0 is not allowed.

7. a. Show by using multiplication that $0 \div 6 = 0$.

b. Is there an integer that equals $0 \div {}^-9$? How do you know?

c. Do you think $0 \div x = 0$ for every integer x *except* 0?

The last three exercises make two things clear.

(1) Division by zero is not allowed.

(2) You can divide 0 by any number *except* 0.

Exercises—4c

1. Suppose you have four downs in a football game. On the first down you gain 3 yards; on the second down you gain 3 yards; on the third down you lose 4 yards; and on the fourth down you also lose 4 yards. You can compute the net yardage in two different ways.

$$(2 \cdot 3) + (2 \cdot {}^-4) = 6 + {}^-8.$$

$$2 \cdot (3 + {}^-4) = 2 \cdot {}^-1.$$

a. Is the net yardage in each case a loss of 2 yards?

b. Is the following sentence true?

$$2 \cdot (3 + {}^-4) = (2 \cdot 3) + (2 \cdot {}^-4).$$

2. Suppose that on Monday you lose 4 marbles; on Tuesday you win 5 marbles; on Wednesday you lose 4 marbles; and on Thursday you again win 5 marbles.

a. Can you compute the net result using the expression below?

$$2({}^-4) + 2(5)$$

b. In what other way might you compute the net result?

3. Make each sentence true.

a. ${}^-2\ ({}^-3 + 7) = ({}^-2 \cdot {}^-3) + ({}^-2 \cdot \_\_\_\_)$.

b. ${}^-4\ (\_\_\_\_ + {}^-6) = ({}^-4 \cdot 5) + ({}^-4 \cdot {}^-6)$.

c. If x, y, and z represent integers, then

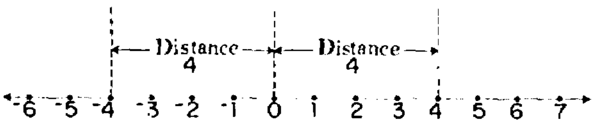
$$x \cdot (y + z) = (\_\_\_\_ \cdot y) + (x \cdot \_\_\_\_).$$

5 Absolute Value

How far from integer to zero place
Is absolute value in any case.
In either direction that you zip
It is the distance of the trip.

Do you like the rhyme? It tells you what "absolute value" means.

Look at the diagram below. Notice that $^{-}4$ and 4 are the same *distance* from zero, even though they are on opposite sides of zero. In each case, the distance from zero is 4 units.



The distance of an integer from zero is called the *absolute value* of the integer. Thus, the absolute value of $^{-}4$ is 4, and the absolute value of 4 is also 4. The two mathematical sentences below provide us with a short way of expressing these ideas. The English sentences on the right tell you how to read the mathematical sentences.

$\lvert 4 \rvert = 4.$	"The absolute value of 4 is equal to 4."
$\lvert ^{-}4 \rvert = 4.$	"The absolute value of $^{-}4$ is equal to 4."

Class Discussion 5a

1. What does absolute value mean according to the rhyme above?

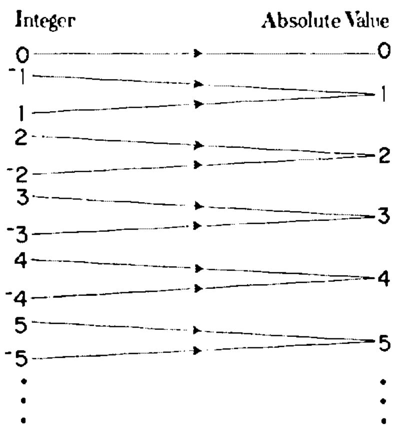
2. Look at the open sentence below.

$$\lvert x \rvert = 4.$$

a. What replacements for x make the sentence true?

b. How many solutions does the open sentence have?

3. Below is a mapping diagram. Each arrow directs you from an integer to the absolute value of the integer.



a. What is the absolute value of $^-10$; 7; $^-13$?
b. Is the absolute value of an integer also an integer?
c. Is the absolute value of an integer ever a negative integer?
d. How many integers can have the same absolute value?
e. Is it true that $|^-4| = |4|$?
f. Is it true that $|0| = 0$?
g. Is zero the absolute value of any integer besides itself?

Summary—5a

1. The absolute value of a positive integer is the integer itself.

   Example: $|5| = 5$.

2. The absolute value of zero is zero.

   Example: $|0| = 0$.

3. The absolute value of a negative integer is the opposite of the negative integer. That is, the absolute value of a negative integer is a positive integer.

   Example: $^-4$ is a negative integer.
   $|^-4| = 4$, which is the opposite of $^-4$.

4. The absolute value of an integer is *never* negative.
5. An integer and its opposite have the same absolute value.

   Example: $|^{-}7| = 7$.

   $|7| = 7$.

   It follows that $|^{-}7| = |7|$.

Exercises—5a

1. Complete each sentence.

a. $|^{-}10| =$ ____. **d.** $|10 - 20| =$ ____.
b. $|2| =$ ____. **e.** $|^{-}10 + 20| =$ ____.
c. $|0| =$ ____. **f.** $|20 - 10| =$ ____.

2. a. $|3 - 5|$ equals ____.
b. $|5 - 3|$ equals ____.
c. Is $|3 - 5|$ less than, equal to, or greater than $|5 - 3|$?

3. In each exercise compare the numbers represented by the two expressions.

a. $|^{-}6 - 4|$ and $|4 - {}^{-}6|$
b. $|8 - {}^{-}5|$ and $|^{-}5 - 8|$
c. $|0 - {}^{-}5|$ and $|^{-}5 - 0|$

4. Which sentence is true if x and y represent integers?

a. $|x - y| < |y - x|$.
b. $|x - y| = |y - x|$.
c. $|x - y| > |y - x|$.

Class Discussion 5b

1. You have learned to add integers by combining gains and losses, and by combining trips along a number line.

a. Use the idea of combining trips along a number line to find the sum $^{-}3 + {}^{-}5$. Maybe you already know a shortcut for adding two negative integers. For example, you may have thought: "I know that 3 + 5 equals 8. Since both trips are west, my answer is $^{-}8$." If you did the problem in this way, you really used the idea of absolute value. To see how absolute value is used to find the sum of two negative integers, complete the sentences below.

b. $|^-3| =$ ___; $|^-5| =$ ___.

c. $|^-3| + |^-5| =$ ___.

d. The opposite of 8 is ___.

2. Use the idea of absolute value to find the sum $^-7 + ^-4$.

a. $|^-7| =$ ___; $|^-4| =$ ___.

b. $|^-7| + |$___$| =$ ___.

c. The opposite of ___ is ___.

d. Now add $^-7$ and $^-4$ using the idea of combining trips along a number line. Does your answer agree with the result you got in exercise 2c?

e. Can you find the sum of two negative integers by adding their absolute values and taking the opposite of the result?

3. Use the idea of absolute value to find each sum.

a. $^-83 + ^-96$

b. $^-862 + ^-284$

4. Do you think the idea of absolute value is useful in finding the sum of two positive integers? Try using the method of exercise 2 to find the sum $6 + 4$.

a. $|6| =$ ___; $|4| =$ ___.

b. $|6| + |4| =$ ___.

c. Does $6 + 4$ equal 10?

d. Is the sum of two positive integers the same as the sum of their absolute values?

5. a. Use the idea of combining trips along a number line to find the sum $^-7 + 3$.

b. Can you find the sum of a positive integer and a negative integer by adding their absolute values?

c. Can you think of a way of adding $^-7$ and 3 without actually counting unit segments on the number line?

6. If your answer to exercise 5c is "yes," you may have reasoned something like this: "I am to go 7 units west and then 3 units east. Since 7 is greater than 3, I will subtract 3 from 7. The

difference is 4. Since 7 is greater than 3, I have gone farther west than east. My answer should be 4 units west, or $^-4$." Let us translate this into "absolute value" language.

English Language	"Absolute Value" Language
I am to go 7 units west and then 3 units east.	$^-7 + 3 = ?$
Since 7 is greater than 3,	Since $\lvert ^-7 \rvert > \lvert 3 \rvert$,
I will subtract 3 from 7. The difference is 4.	$\lvert ^-7 \rvert - \lvert 3 \rvert = 4$.
Since 7 is greater than 3,	Since $\lvert ^-7 \rvert > \lvert 3 \rvert$,
I have gone farther west than east.	
My answer should be 4 units west.	$^-7 + 3 = {}^-4$.

a. Complete each sentence on the right.

English Language	"Absolute Value" Language
I am to go 6 units west and then 15 units east.	____ + ____ = ?
Since 15 is greater than 6,	Since |____| > |____|,
I will subtract 6 from 15. The difference is 9.	|____| − |____| = ____ .
Since 15 is greater than 6,	Since |____| > |____|,
I have gone farther east than west.	
My answer should be 9 units east.	____ + ____ = ____ .

b. Use the idea of absolute value to find the sum $16 + {}^-28$.

c. Use the idea of absolute value to find the sum $^-110 + 516$.

Summary—5b

1. To find the sum of two positive integers, find the sum of their absolute values.
2. To find the sum of two negative integers, find the sum of their absolute values and take the opposite of the result.
3. To find the sum of a positive integer and a negative integer, begin by subtracting the lesser absolute value from the greater absolute value. If the positive integer has the greater absolute value, the sum is positive. If the negative integer has the greater absolute value, the sum is negative.

Exercises—5b

1. Find each sum, using the idea of absolute value.

a. $^-6 + 7$	e. $12 + {}^-7$
b. $^-13 + 9$	f. $35 + {}^-78$
c. $^-9 + {}^-3$	g. $^-10 + {}^-46$
d. $84 + {}^-27$	h. $^-67 + {}^-28$

2. a. Suppose both x and y represent negative integers, then $x + y =$ the ____ of $|x| + |y|$.
   b. If both x and y represent positive integers, then $x + y = |x| +$ ____.
3. a. Does $|5 + 4|$ equal $|5| + |4|$?
   b. Does the absolute value of the sum of two positive integers equal the sum of their absolute values?
4. a. Does $|^-7 + {}^-8| = |^-7| + |^-8|$?
   b. Does the absolute value of the sum of two negative integers equal the sum of their absolute values?
5. a. Suppose both x and y represent negative integers. Is the following sentence true?

$$|x + y| = |x| + |y|.$$

   b. Is this sentence true if x and y both represent positive integers?

Exercises—5c

The idea of absolute value can be used to find the product of two integers. Use the rules below as a guide.

(1) First find the product of the absolute values of the two integers.

(2) If the two integers are on the same side of zero, the product is positive.

(3) If the two integers are on opposite sides of zero, the product is negative.

(4) If either integer is zero, the product is zero.

1. Find the product $^-3 \cdot 5$.
   a. $|^-3| \cdot |5| =$ ____.
   b. Are $^-3$ and 5 on opposite sides of zero?
   c. Does $^-3 \cdot 5$ equal the opposite of $|^-3| \cdot |5|$?
   d. $^-3 \cdot 5 =$ ____.
2. Find the product $^-4 \cdot {}^-7$.
   a. $|^-4| \cdot |^-7| =$ ____.
   b. Are $^-4$ and $^-7$ on the same side of zero?
   c. Does $^-4 \cdot {}^-7 = |^-4| \cdot |^-7|$?
   d. $^-4 \cdot {}^-7 =$ ____.
3. a. Are 5 and 6 on the same side of zero?
   b. Does $5 \cdot 6 = |5| \cdot |6|$?
4. Find each product.
   a. $^-10 \cdot 0$
   b. $0 \cdot {}^-7$
5. Use the idea of absolute value to find each product.
   a. $^-15 \cdot 14$
   b. $^-19 \cdot {}^-36$
   c. $100 \cdot {}^-100$
   d. $(^-4 \cdot {}^-6) \cdot {}^-10$
   e. $[(^-7 \cdot {}^-2) \cdot {}^-6] \cdot {}^-1$
6. Which sentence is false?
   a. $|^-5| \cdot |^-7| = |^-5 \cdot {}^-7|$.
   b. $|6| \cdot |^-9| = |6 \cdot {}^-9|$.
   c. $|^-3| \cdot |8| = |^-3 \cdot 8|$.
7. Suppose x and y represent integers. Are there replacements for x and y that make the following sentence false?

$$|x| \cdot |y| = |x \cdot y|.$$

Summary—Sc

The product of two positive integers is positive.

The product of two negative integers is positive.

The product of a positive integer and a negative integer in either order is negative.

The product of any integer and zero is zero.

The absolute value of the product of two integers is the product of the absolute values of the integers. That is, if x and y represent integers, then

$$|x \cdot y| = |x| \cdot |y|.$$

Summarized below are statements of the properties of multiplication of integers.

1. The product of two integers is an integer.

   If x and y represent integers, then $x \cdot y$ represents an integer.
2. The order in which two integers are multiplied does not affect the product.

   If x and y represent integers, then $x \cdot y = y \cdot x$.
3. The way in which three integers are grouped when multiplying does not affect the product.

   If x, y, and z represent integers, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
4. The product of any given integer and 1 is the given integer.

   If x represents an integer, then $1 \cdot x = x \cdot 1 = x$.
5. The product of any given integer and zero is zero.

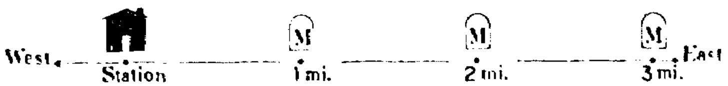
   If x represents an integer, then $x \cdot 0 = 0 \cdot x = 0$.
6. Multiplication of integers distributes over addition of integers.

   If x, y, and z represent integers, then

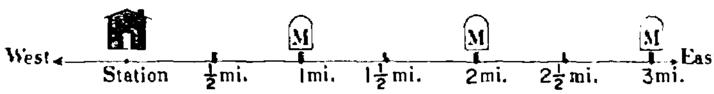
$$x \cdot (y + z) = (x \cdot y) + (x \cdot z).$$

6 Traveling on the Rational Railroad

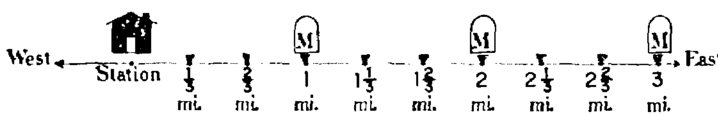
The Rational Railroad gives the public unusual service. All trains stop at the station. Train No. 1 stops at every mile marker. The diagram shows several mile markers east of the station.



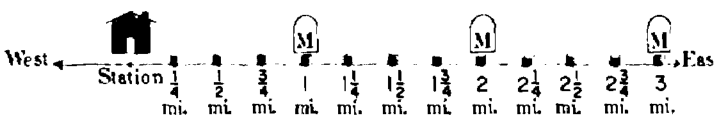
Train No. 2 stops at every mile marker and halfway between every pair of mile markers. The distance between stops is $\frac{1}{2}$ mile. The red marks in the diagram below indicate stops for Train No. 2.



Train No. 3 stops at every mile marker and also at points that are $\frac{1}{3}$ and $\frac{2}{3}$ of the distance from one mile marker to the next. The distance between stops is $\frac{1}{3}$ mile. The red arrowheads in the diagram below indicate stops for Train No. 3.

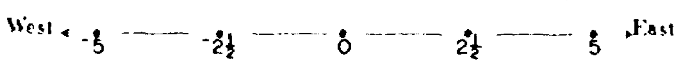


Train No. 4 stops at every mile marker and at points that are $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the distance from one mile marker to the next. The distance between stops is $\frac{1}{4}$ mile. The red blocks in the diagram below show stops of Train No. 4.



Suppose that the pattern of stops for Train Nos. 1, 2, 3, and 4 is continued for each additional train that is put into service, and that the supply of trains is unlimited. The distance between stops for Train No. 5 will be $\frac{1}{5}$ mile. The distance between stops for Train No. 6 will be $\frac{1}{6}$ mile, and so on. You can see that if the Rational Railroad puts more and more trains into service, stops become closer and closer.

For any stop east of the station there is exactly *one* stop that is the same distance west of the station. See the diagram below.



The diagram shows that *numbers* can be assigned to stops on the Rational Railroad. Let us call the numbers that are assigned to stops east of the station *positive rational numbers,* and those that are assigned to stops west of the station *negative rational numbers.* The number assigned to the station stop is the *rational number zero.*

The set of numbers that can be assigned to the set of all stops on the Rational Railroad is called the *set of rational numbers.* Therefore, you can think of the Rational Railroad as a number line. As the diagram above clearly shows, the integers belong to the set of rational numbers. In other words, the set of integers is a subset of the set of rational numbers.

Class Discussion 6a

1. Make a copy of the number line below and assign numbers to all points indicated by dots.

2. Use the idea of successive trips along a number line to find each sum.

   a. $\frac{1}{4} + \frac{1}{2}$ e. $3\frac{1}{2} + {}^{-}2\frac{1}{4}$

   b. ${}^{-}\frac{1}{4} + {}^{-}\frac{1}{4}$ f. ${}^{-}3\frac{1}{2} + 2\frac{1}{4}$

   c. ${}^{-}\frac{1}{2} + \frac{1}{4}$ g. $3\frac{1}{2} + {}^{-}2\frac{1}{4}$

   d. $2\frac{1}{2} + 1\frac{1}{4}$ h. ${}^{-}1\frac{1}{2} + {}^{-}2\frac{1}{4}$

3. What number is the opposite of each of the following?

   a. $\frac{1}{4}$ d. $2\frac{1}{4}$ g. ${}^{-}3\frac{1}{2}$

   b. ${}^{-}\frac{1}{2}$ e. ${}^{-}2\frac{1}{4}$ h. ${}^{-}1\frac{1}{4}$

   c. ${}^{-}\frac{3}{4}$ f. $1\frac{1}{2}$ i. $3\frac{1}{4}$

4. For each number listed in exercise 3, find the sum of the number and its opposite.

5. To subtract one integer from another, you added the opposite of the second integer to the first. Use this idea to find the difference in each exercise. Use a number line if you wish.

a. $\frac{1}{4} - \frac{1}{2}$	**e.** $^{-}3\frac{1}{2} - {}^{-}2\frac{1}{4}$
b. $\frac{1}{4} - {}^{-}\frac{1}{2}$	**f.** $3\frac{1}{2} - {}^{-}2\frac{1}{4}$
c. $\frac{1}{2} - {}^{-}\frac{1}{4}$	**g.** $^{-}1\frac{1}{2} - {}^{-}3\frac{1}{2}$
d. $3\frac{1}{2} - 2\frac{1}{4}$	

Summary—6a

Below is a summary of the properties of addition of rational numbers.

1. The sum of two rational numbers is again a rational number.
   If x and y represent rational numbers, then $x + y$ represents a rational number.
2. The order in which two rational numbers are added does not affect the sum.
   If x and y represent rational numbers, then $x + y = y + x$.
3. The way in which three rational numbers are grouped when adding does not affect the sum.
   If x, y, and z represent rational numbers, then $x + (y + z) = (x + y) + z$.
4. The sum of any given rational number and zero is the given rational number.
   If x represents a rational number, then $x + 0 = 0 + x = x$.
5. Every rational number has exactly one opposite. The sum of a rational number and its opposite is zero.

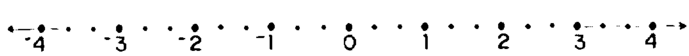
If x represents a rational number, then there is exactly one rational number y such that $x + y = 0$.

Zero is its own opposite: that is, $0 + 0 = 0$.

If x and y represent rational numbers, then the difference $x - y$ equals $x + {}^-y$. The symbol ${}^-y$ may either be read "the opposite of y" or "the negative of y."

Exercises—6a

1. Make a copy of the number line below and assign numbers to all points indicated by dots.



a. Find each sum. Use the number line above if you wish.

$^-1\frac{1}{3} + 2\frac{2}{3}$ $\quad$ $4 + {}^-3\frac{1}{3}$

$^-2 + {}^-1\frac{2}{3}$ $\quad$ $^-4\frac{1}{3} + 1$

b. Find each difference. Use the number line above if you wish.

$2\frac{2}{3} - {}^-1$ $\quad$ $0 - {}^-\frac{2}{3}$

$^-3\frac{1}{3} - {}^-\frac{2}{3}$ $\quad$ $^-2 - 1\frac{1}{3}$

2. Make a copy of the number line below and assign numbers to all points indicated by dots.

a. Find each sum.

$\frac{1}{7} + {}^-\frac{4}{7}$ $\quad$ $1\frac{3}{7} + {}^-1\frac{4}{7}$

$^-1\frac{2}{7} + {}^-\frac{5}{7}$ $\quad$ $^-1 + {}^-2\frac{6}{7}$

b. Find each difference.

$\frac{1}{7} - \frac{^-4}{7}$ $1\frac{3}{7} - {}^-1\frac{1}{7}$ ${}^-1\frac{2}{7} - \frac{^-5}{7}$ ${}^-1 - {}^-2\frac{6}{7}$

3. Recall that the absolute value of an integer is defined as the distance of the integer from zero. The absolute value of a rational number is defined in the same way.

a. $\left|\frac{2}{7}\right| = $ ___ ; $\left|\frac{^-5}{7}\right| = $ ___ .

b. Is it true that $\left|\frac{2}{7}\right| < \left|\frac{^-5}{7}\right|$?

c. Does $\left|\frac{^-5}{7}\right| - \left|\frac{2}{7}\right| = \frac{3}{7}$?

d. $\frac{2}{7} + \frac{^-5}{7} = $ ___ .

4. Use the idea of absolute value to find the sum $\frac{3}{4} + \frac{^-1}{4}$.

a. $\left|\frac{3}{4}\right| = $ ___ ; $\left|\frac{^-1}{4}\right| = $ ___ .

b. Is it true that $\left|\frac{3}{4}\right| > \left|\frac{^-1}{4}\right|$?

c. Does $\left|\frac{3}{4}\right| - \left|\frac{^-1}{4}\right| = \frac{1}{2}$?

d. $\frac{3}{4} + \frac{^-1}{4} = $ ___ .

5. Use the idea of absolute value to find the sum $\frac{^-1}{5} + \frac{^-2}{5}$.

a. $\left|\frac{^-1}{5}\right| = $ ___ ; $\left|\frac{^-2}{5}\right| = $ ___ .

b. $\frac{^-1}{5} + \frac{^-2}{5}$ equals the ___ of $\left|\frac{^-1}{5}\right| + \left|\frac{^-2}{5}\right|$.

c. $\frac{^-1}{5} + \frac{^-2}{5} = $ ___ .

6. a. Is it true that $\frac{1}{3} + \frac{2}{3} = \left|\frac{^-1}{3}\right| + \left|\frac{^-2}{3}\right|$?

b. $\frac{1}{3} + \frac{2}{3} = $ ___ .

7. Use the idea of absolute value to find each sum.

a. $^{-}\frac{3}{7} + 1\frac{5}{7}$ **b.** $\frac{3}{4} + {}^{-}1\frac{1}{4}$ **c.** $^{-}\frac{2}{5} + \frac{3}{5}$ **d.** $^{-}\frac{3}{8} + {}^{-}\frac{5}{8}$

Class Discussion 6b

To find the product of two rational numbers, use the rules below as a guide.

(1) First, find the product of the absolute values of the two rational numbers.

(2) If the two rational numbers are on the same side of zero, the product is positive.

(3) If the two rational numbers are on opposite sides of zero, the product is negative.

(4) If either rational number is zero, the product is zero

1. Find the product $^{-}\frac{2}{5} \cdot \frac{3}{4}$. Use the steps below as a guide.

a. First, find the product of the absolute values of $^{-}\frac{2}{5}$ and $\frac{3}{4}$.

$$\left|{}^{-}\frac{2}{5}\right| \cdot \left|\frac{3}{4}\right| = \frac{2}{5} \cdot \frac{3}{4} = \underline{\quad\quad}.$$

$\left(\text{Recall that } \frac{2}{5} \cdot \frac{3}{4} = \frac{2 \cdot 3}{5 \cdot 4} = \frac{6}{20} = \frac{3}{10}.\right)$

b. Are $^{-}\frac{2}{5}$ and $\frac{3}{4}$ on opposite sides of zero?

c. Does $^{-}\frac{2}{5} \cdot \frac{3}{4}$ equal a negative number?

d. $^{-}\frac{2}{5} \cdot \frac{3}{4} = \underline{\quad\quad}.$

2. Find the product $^{-}\frac{2}{3} \cdot {}^{-}\frac{1}{5}$. Use the steps below as a guide.

a. $\left|{}^{-}\frac{2}{3}\right| \cdot \left|{}^{-}\frac{1}{5}\right| = \underline{\quad\quad}.$

b. Are $\frac{^-2}{3}$ and $\frac{^-1}{5}$ on the same side of zero?

c. Does $\frac{^-2}{3} \cdot \frac{^-1}{5}$ equal $\left|\frac{^-2}{3}\right| \cdot \left|\frac{^-1}{5}\right|$?

d. $\frac{^-2}{3} \cdot \frac{^-1}{5} = $ ——.

3. a. Does $\frac{3}{4} \cdot \frac{4}{5} = \left|\frac{3}{4}\right| \cdot \left|\frac{4}{5}\right|$?

b. $\frac{3}{4} \cdot \frac{4}{5} = $ ——.

4. Find the products.

a. $\frac{1}{2} \cdot \frac{3}{5}$ d. $\frac{^-3}{5} \cdot \frac{2}{7}$

b. $\frac{3}{5} \cdot \frac{2}{7}$ e. $\frac{^-3}{5} \cdot \frac{^-2}{7}$

c. $\frac{3}{5} \cdot \frac{^-2}{7}$ f. $\frac{^-2}{5} \cdot \frac{3}{8}$

5. Find the products.

a. $\frac{3}{5} \cdot \frac{5}{3}$ d. $^-1\frac{1}{2} \cdot \frac{^-2}{3}$

b. $\frac{^-3}{5} \cdot \frac{^-5}{3}$ e. $2\frac{1}{3} \cdot \frac{3}{7}$

c. $\frac{^-7}{2} \cdot \frac{^-2}{7}$ f. $^-3 \cdot \frac{^-1}{3}$

6. a. Does each product in exercise 5 equal 1?

b. Find three other pairs of rational numbers such that the product of each pair is 1.

c. If the product of two numbers is 1, each number is the *reciprocal* of the other. What is the reciprocal of $\frac{^-3}{4}$; $\frac{7}{10}$; $\frac{^-11}{2}$?

d. Is there a rational number x that makes the following sentence true?

$$x \cdot 0 = 1.$$

e. Does zero have a reciprocal?

7. In working with integers you may have discovered that the quotient of two integers is *not always* an integer.
   a. Does $7 \div 2$ give you an integer for an answer?
   b. Here is another way of asking the same question: Is there an integer x for which it is true that $7 \div 2 = x$?

8. In working with rational numbers, division (except by zero) is always possible. Let's explore how to find the quotient of two rational numbers.
   a. Division and multiplication are related. For example, to divide 42 by 7 you need to find a number x such that $x \cdot 7 = 42$. Do the two open sentences below have the same solution? How are the two sentences related?

$$42 \div 7 = x.$$

$$x \cdot 7 = 42.$$

   b. Look at the next two sentences.

$$\frac{3}{10} \div \frac{3}{5} = \frac{1}{2}.$$

$$\frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}.$$

   Is the second sentence true? Is the first sentence true?

9. Look at the last two sentences again, and pretend you don't know what the quotient is when you divide $\frac{3}{10}$ by $\frac{3}{5}$.

$$\frac{3}{10} \div \frac{3}{5} = x.$$

$$x \cdot \frac{3}{5} = \frac{3}{10}.$$

   Do the two open sentences have the same solution?
   a. The steps below show how to solve the open sentence $x \cdot \frac{3}{5} = \frac{3}{10}$. In the first step each side is multiplied by the *reciprocal* of $\frac{3}{5}$. Why can you do this?

(1) Multiply both sides by $\frac{3}{5}$. $\left(x \cdot \frac{3}{5}\right) \cdot \frac{5}{3} = \frac{3}{10} \cdot \frac{5}{3}$.

(2) Group $\frac{3}{5}$ with $\frac{5}{3}$. $x \cdot \left(\frac{3}{5} \cdot \frac{5}{3}\right) = \frac{3}{10} \cdot \frac{5}{3}$.

(3) Find the product $\frac{3}{5} \cdot \frac{5}{3}$. $x \cdot 1 = \frac{3}{10} \cdot \frac{5}{3}$.

(4) Replace $x \cdot 1$ by x. $x = \frac{3}{10} \cdot \frac{5}{3}$.

(5) Find the product $\frac{3}{10} \cdot \frac{5}{3}$. $x = \frac{1}{2}$.

b. The rational number $\frac{1}{2}$ is the solution of

$$x \cdot \frac{3}{5} = \frac{3}{10}.$$

c. Is $\frac{1}{2}$ also the solution of the related division sentence?

$$\frac{3}{10} \div \frac{3}{5} = x.$$

10. Solve the open sentence by solving the related multiplication sentence.

$$\frac{{}^{-}3}{7} \div \frac{2}{3} = x$$

John and David were doing their homework. The conversation went something like this:

John: Dave, suppose you had to find the quotient $\frac{8}{21} \div \frac{2}{3}$. Why couldn't you just divide 8 by 2 and 21 by 3? Since 8 divided by 2 is 4 and 21 divided by 3 is 7, you'd get $\frac{4}{7}$ for an answer.

David: Well, let's see if this checks. $\frac{4}{7} \cdot \frac{2}{3} = \frac{8}{21}$, and it does work in this case. But does the scheme work in all cases?

John: Here's an example in which it doesn't work. Find the quotient $\frac{3}{5} \div \frac{2}{7}$. You can't divide 3 by 2, and you can't divide 5 by 7.

David: Don't give up so easily. You know that you can multiply the numerator and denominator of $\frac{3}{5}$ by the same nonzero whole number and still have the same rational number. So multiply the numerator and denominator both by 2. This gives you $\frac{6}{10}$, and you can divide 6 by 2.

John: Yes, that's great, but you can't divide 10 by 7.

David: All right, just multiply both the numerator and denominator of $\frac{6}{10}$ by 7.

John: Let me get this down on paper so I can remember how it's done. We now have $\frac{3 \cdot 2 \cdot 7}{5 \cdot 2 \cdot 7} \div \frac{2}{7}$, and canceling the 2's in the numerators and the 7's in the denominators we get $\frac{3 \cdot 2 \cdot 7}{5 \cdot 2 \cdot 7} \div \frac{2}{7} = \frac{3 \cdot 7}{5 \cdot 2}$.

David: Look, John! You could have found the same thing right away if you had multiplied $\frac{3}{5}$ by the reciprocal of $\frac{2}{7}$. This is what you have as a result.

Now you know why people started multiplying the dividend by the reciprocal of the divisor when dividing rational numbers.

Summary—6b

Summarized below are properties of multiplication of rational numbers.

1. The product of two rational numbers is again a rational number.

   If x and y represent rational numbers, then $x \cdot y$ represents a rational number.

2. The order in which two rational numbers are multiplied does not affect the product.

   If x and y represent rational numbers, then $x \cdot y = y \cdot x$.

3. The way in which three rational numbers are grouped when multiplying does not affect the product.

   If x, y, and z represent rational numbers, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

4. The product of any given rational number and 1 is the given rational number.

   If x represents a rational number, then $x \cdot 1 = 1 \cdot x = x$.

5. The product of any given rational number and zero is zero.

   If x represents a rational number, then $x \cdot 0 = 0 \cdot x = 0$.

6. Multiplication of rational numbers distributes over addition of rational numbers.

   If x, y, and z represent rational numbers, then $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

7. For each rational number x, except zero, there is exactly one rational number y (called the reciprocal of x) such that $x \cdot y = 1$.

Exercises—6

1. Find the products.

   a. $\frac{2}{5} \cdot \frac{^-3}{5}$ c. $\frac{2}{5} \cdot \frac{^-5}{3}$

   b. $\frac{^-1}{3} \cdot \frac{3}{4}$ d. $\frac{^-2}{5} \cdot \frac{^-3}{8}$

2. Give the reciprocal of each number.

   a. $\frac{7}{8}$ d. $\frac{3}{100}$

   b. $\frac{^-9}{4}$ e. 10

   c. $^-1$

3. Find each quotient.

a. $\frac{^-2}{5} \div \frac{3}{8}$ d. $\frac{6}{1} \div \frac{^-1}{3}$

b. $\frac{^-3}{7} \div \frac{^-3}{7}$ e. $\frac{2}{3} \div \frac{^-4}{1}$

c. $\frac{^-5}{8} \div \frac{5}{8}$ f. $\frac{^-7}{1} \div \frac{^-2}{3}$

General Summary

Before you learned about negative numbers, you used only that part of the number line which includes point 0 and all points to the right of zero. In this unit you learned that numbers can be assigned to points that are to the left of zero. Numbers to the right of zero are called positive numbers, and numbers to the left of zero are called negative numbers. Zero is neither positive nor negative.

When you had only positive numbers to work with, you were not able to subtract a larger number from a smaller number. However, when you have both positive and negative numbers to work with, subtraction of one number from another in either order is always possible.

For any number on one side of zero there is exactly one number on the other side of zero such that the sum of the two numbers is zero. Each of the two numbers is called the opposite of the other. Because every number on the number line has an opposite, you can change every subtraction problem to an addition problem. That is, to subtract a number, add its opposite. This is convenient, because you can always change the order in which you add two numbers but not the order in which you subtract.

Both integers and rational numbers that are not integers can be assigned to points in the number line. While consecutive integers are usually assigned to points that are one unit apart, other rational numbers can be assigned to points that are as close as you please. Between any two rational numbers there is always another rational number.

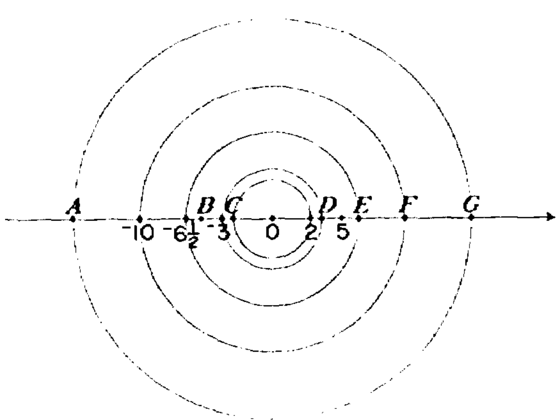
With so many rational numbers so close together, it may surprise you that there are some points in the number line that cannot be associated with rational numbers. In fact, there are an infinite number of these unused points! If you study more mathematics, you will learn that these points are associated with numbers called irrational numbers. The union of the set of rational numbers and the set of irrational numbers is called the set of *real* numbers. Thus a real number can be assigned to every point in the number line, and every point in the number line can be matched with one real number.

Review Exercises

1. Complete each sentence, using whichever symbol, =, <, >, makes the sentence true.

a. 3 ____ $^-3$. **d.** $^-6$ ____ 2.
b. 0 ____ $^-8$. **e.** $|^-6|$ ____ $|2|$.
c. $^-3$ ____ $^-7$. **f.** $|^-3|$ ____ $|3|$.

2. Each circle shown below has point 0 as center. What numbers should be assigned to the points labeled by letters?



3. Find each sum.

a. $^-2 + {}^-7$
b. $^-9 + 6$
c. $^-98 + {}^-47$
d. $136 + 62$
e. $^-8 + 3$
f. $^-3 + 8$
g. $|{}^-3| + |8|$
h. $^-8 + {}^-3$

4. Use the idea of opposites to change each subtraction to an addition.

a. $5 - 8 = 5 +$ ____.
b. $5 - {}^-8 = 5 +$ ____.
c. $^-8 - 2 =$ ____ $+$ ____.
d. $^-6 - {}^-3 =$ ____ $+$ ____.
e. $0 - 7 =$ ____ $+$ ____.

5. Find each difference.

a. $3 - 8$
b. $9 - {}^-6$
c. $^-12 - {}^-7$
d. $^-3 - 15$
e. $0 - {}^-8$
f. $^-5 - {}^-5$
g. $^-5 - 5$

6. Find each product.

a. $8 \cdot {}^-4$
b. $^-7 \cdot {}^-6$
c. $3 \cdot 12$
d. $^-1 \cdot 24$
e. $^-5 \cdot 5$
f. $0 \cdot {}^-7$

7. For each division sentence, complete the related multiplication sentence.

a. $12 \div 4 = 3$; $3 \cdot$ ____ $= 12$.
b. $^-12 \div 4 = {}^-3$; ____ $\cdot 4 =$ ____.
c. $^-16 \div {}^-8 = x$; ____ $\cdot$ ____ $=$ ____.
d. $20 \div {}^-5 = x$; ____ $\cdot$ ____ $=$ ____.
e. $^-35 \div 7 = x$; ____ $\cdot$ ____ $=$ ____.

8. Find each quotient.

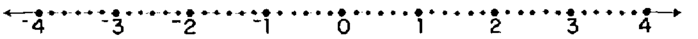
a. $^-16 \div {}^-8$
b. $20 \div {}^-5$
c. $^-35 \div 7$
d. $^-28 \div {}^-7$
e. $72 \div {}^-8$
f. $0 \div {}^-2$

9. Use the number line below to find each sum.

a. $1\frac{1}{3} + {}^-2\frac{2}{3}$ **b.** ${}^-1\frac{5}{6} + {}^-2\frac{5}{6}$ **c.** ${}^-\frac{5}{6} + 3\frac{1}{6}$ **d.** $2\frac{1}{3} + 3\frac{2}{3}$

10. Use the number line below to find each difference.

a. $^-1 - 1\frac{1}{3}$ **b.** $2 - \frac{^-5}{6}$ **c.** $^-1\frac{1}{6} - {}^-1\frac{1}{6}$



11. Make each sentence true.

a. $5 \cdot {}^-6 = {}^-6 \cdot$ ____.
b. $^-8 + ({}^-2 + 6) = ({}^-8 +$ ____$) + 6$.
c. $^-13 \cdot$ ____ $= 0$.
d. $9 - {}^-2 = 9 +$ ____.
e. $5 \cdot ({}^-3 + 3) = (5 \cdot {}^-3) + (5 \cdot$ ____$)$.
f. $^-7 +$ ____ $= {}^-3 + 7$.
g. $^-12 +$ ____ $= 0$.
h. $|{}^-5 + 2| =$ ____.
i. ____ $\cdot {}^-18 = {}^-18$.
j. $({}^-1 \cdot 7) \cdot {}^-4 =$ ____ $\cdot (7 \cdot {}^-4)$.

12. Which sentences are true and which are false?

a. The sum of a number and its opposite is 0.
b. The absolute value of a negative number is always its opposite.
c. A number can be its own opposite.
d. When subtracting one number from another, the order does not matter.
e. The sum of a positive number and a negative number is always negative.
f. The product of two negative numbers is always positive.
g. The sum of two negative numbers is always negative.
h. A negative number may be greater than a positive number.

13. For each exercise, decide whether or not it is possible to replace x and y by two integers that make the sentence true. Will any of the sentences be true for every pair of integers you might select?

a. $|x| + |y| = |x + y|$. **d.** $|x| - |y| = |x - y|$.
b. $|x| + |y| > |x + y|$. **e.** $|x| - |y| < |x - y|$.
c. $|x| + |y| < |x + y|$. **f.** $|x| - |y| > |x - y|$.